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ON THE DESCENDING SERIES FOR BESSEL'S FUNCTIONS OF BOTH KINDS.

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1. It has been shown in the Annals* that Bessel's equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\,y = 0 \tag{1}$$

has for complete primitive, when n is fractional,

$$y = AJ_n(x) + BJ_{-n}(x)\,, \tag{2}$$

and when n is integral

$$y = AJ_n(x) + BK_n(x)\,, \tag{3}$$

where

$$J_n(x) = \left[\frac{x}{2}\right]^n \sum_{p=0}^{p=\infty} \frac{(-1)^p}{\Gamma(n+p+1)\,\Gamma(p+1)} \left[\frac{x}{2}\right]^{2p}; \tag{4}$$

and

$$2K_n(x) = \frac{dJ_n(x)}{dn} + (-1)^{n+1}\frac{dJ_{-n}(x)}{dn}\,. \tag{5}$$

Although this series for $J_n(x)$, and that given on p. 88 for $K_n(x)$, are convergent for all values of x, yet when x is large the convergence is so slow as to render the series unavailable for numerical computation.

In such a case the descending series are readily applicable; and it is proposed to derive from equation (1) a general solution in negative powers of x, involving two arbitrary constants, and then to determine these constants for the cases of $J_n(x)$, $K_n(x)$, respectively.

2. *General solution in negative powers.*

In equation (1) put $y = x^m u$, then

$$x^2 \frac{d^2u}{dx^2} + (2m+1)\,x\frac{du}{dx} + (x^2 + m^2 - n^2)\,u = 0\,.$$

To remove the second term, let $m = -\frac{1}{2}$, then

$$\frac{d^2u}{dx^2} + u = (n^2 - \tfrac{1}{4})\frac{u}{x^2}\,, \tag{6}$$

in which $u = x^{\frac{1}{2}}y$. If the right hand member were zero, the solution would be

$$u = Ae^{ix} + Be^{-ix}\,. \tag{7}$$

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* Vol. VI, No. 4, p. 86. "On Bessel's Functions of the Second Kind," by Dr. Maxime Bôcher.

Annals of Mathematics, Vol. VIII, No. 2.

Substitute in (6), regarding A, B as functions of x, and identify separately the coefficients of e^{ix}, e^{-ix}; then

$$\left.\begin{aligned} \frac{d^2A}{dx^2} + 2i\frac{dA}{dx} &= (n^2 - \tfrac{1}{4})\frac{A}{x^2}, \\ \frac{d^2B}{dx^2} - 2i\frac{dB}{dx} &= (n^2 - \tfrac{1}{4})\frac{B}{x^2}. \end{aligned}\right\} \tag{8}$$

Put $ix = z$; then

$$\frac{d^2A}{dz^2} + 2\frac{dA}{dz} = (n^2 - \tfrac{1}{4})\frac{A}{z^2}.$$

Let
$$A = A_0 + A_1 z^{-1} + \ldots + A_s z^{-s} + \ldots, \tag{9}$$

and substitute and equate coefficients of z^{-s-2};

$$\therefore \quad A_{s+1} = A_s \frac{s(s+1) + \frac{1}{4} - n^2}{2(s+1)} = A_s \frac{(2s+1)^2 - 4n^2}{8(s+1)}.$$

Substitute in (9) for A_1, A_2, ..., and replace z by ix; then

similarly,
$$\left.\begin{aligned} A &= A_0\left[1 + \frac{1^2 - 4n^2}{1 \,.\, 8ix} + \frac{(1^2 - 4n^2)(3^2 - 4n^2)}{1 \,.\, 2 \,.\, (8ix)^2} + \ldots\right], \\ B &= B_0\left[1 - \frac{1^2 - 4n^2}{1 \,.\, 8ix} + \frac{(1^2 - 4n^2)(3^2 - 4n^2)}{1 \,.\, 2 \,.\, (8ix)^2} - \ldots\right]; \end{aligned}\right\} \tag{10}$$

$\therefore$, from (7),

$$x^{\frac{1}{2}}y = A_0 e^{ix}\left[1 + \frac{1^2 - 4n^2}{1 \,.\, 8ix} + \ldots\right] + B_0 e^{-ix}\left[1 - \frac{1^2 - 4n^2}{1 \,.\, 8ix} + \ldots\right], \tag{11}$$

in which A_0, B_0 are arbitrary constants.

To make this expression real in form, put

$$A_0 = \tfrac{1}{2} P e^{-\alpha i}, \quad B_0 = \tfrac{1}{2} P e^{\alpha i},$$

and then replace imaginary exponentials by trigonometric equivalents;

$$\begin{aligned} \therefore\ x^{\frac{1}{2}}y = {} & P\cos(x - \alpha)\left[1 - \frac{(1^2 - 4n^2)(3^2 - 4n^2)}{1 \,.\, 2 \,.\, (8x)^2} + \ldots\right] \\ & + P\sin(x - \alpha)\left[\frac{1^2 - 4n^2}{1 \,.\, 8x} - \frac{(1^2 - 4n^2)(3^2 - 4n^2)(5^2 - 4n^2)}{1 \,.\, 2 \,.\, 3 \,.\, (8x)^3} + \ldots\right]. \end{aligned} \tag{12}$$

This expression for y is a general solution of equation (1), involving the two arbitrary constants P, α.

3. *To determine the constants P, a, in the case of $J_n(x)$.*

In the first place it may be verified that the ascending series for $J_n(x)$ is equivalent to the definite integral

$$\frac{1}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})} \cdot \left[\frac{x}{2}\right]^n \int_0^\pi \cos(x\cos\varphi)\sin^{2n}\varphi\,d\varphi\,; \tag{13}$$

for the coefficient of x^{2p} under the integral sign is

$$\frac{(-1)^p}{(2p)!}\int_0^\pi \cos^{2p}\varphi\sin^{2n}\varphi\,d\varphi\,,$$

which equals[*]

$$\frac{(-1)^p}{(2p)!}\,\frac{\Gamma(p+\frac{1}{2})\,\Gamma(n+\frac{1}{2})}{\Gamma(n+p+1)}\,.$$

Then writing $\Gamma(p+\frac{1}{2}) = (p-\frac{1}{2})(p-\frac{3}{2})\ldots\frac{1}{2}\,\Gamma(\frac{1}{2}) = \frac{(2p)!}{p!\,2^{2p}}\sqrt{\pi}$, and replacing $p!$ by $\Gamma(p+1)$, the identity of the general term of (13) with that of (4) will be apparent.

It is next proposed to determine the constants P, a, by comparing the forms taken by (12) and (13) when x is very large.[†] For convenience, let x be of the form $2r\pi$, where r is a large integer, then

$$\cos(x\cos\varphi) = \cos(2x\cos^2\tfrac{1}{2}\,\varphi - x) = \cos(2x\cos^2\tfrac{1}{2}\,\varphi)\,,$$

and the integral takes the form

$$\int_0^\pi \cos(2x\cos^2\tfrac{1}{2}\,\varphi)\sin^{2n}\varphi\,d\varphi\,,$$

which, by the substitution $2x\cos^2\frac{1}{2}\,\varphi = t$, becomes

$$\frac{2^{n-\frac{1}{2}}}{x^{n+\frac{1}{2}}}\int_0^{2x}\cos t\,.\,t^{n-\frac{1}{2}}\,(1-\tfrac{1}{2}\,tx^{-1})^{n-\frac{1}{2}}dt\,.$$

Taking this integral between 0 and x, and then between x and $2x$, and transforming the latter part so that its limits become 0 and x, it will be seen that the two parts are equal, and the integral becomes

$$\frac{2^{n+\frac{1}{2}}}{x^{n+\frac{1}{2}}}\int_0^{x}\cos t\,.\,t^{n-\frac{1}{2}}(1-\tfrac{1}{2}\,tx^{-1})^{n-\frac{1}{2}}dt;$$

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[*] Williamson's Integral Calculus, Art. 122.

[†] The form taken by (13) when x is large is suggested by Art. 406, Todhunter's Functions.

whence (13) ultimately takes the form

$$J_n(x) = \left[\frac{2}{\pi x}\right]^{\frac{1}{2}} \frac{1}{\Gamma(n+\frac{1}{2})}\int_0^\infty \cos t \,.\, t^{n-\frac{1}{2}} dt\,,$$

which reduces to

$$\left[\frac{2}{\pi x}\right]^{\frac{1}{2}} \cos \tfrac{1}{2}(n+\tfrac{1}{2})\pi\,.^*$$

But the general solution for y in (12) becomes, on the same supposition (viz. that x is large and of the form $2r\pi$),

$$y = \frac{P}{\sqrt{x}}\cos(x-a) = \frac{P}{\sqrt{x}}\cos a\,;$$

whence it follows that

$$P = \left[\frac{2}{\pi}\right]^{\frac{1}{2}}, \qquad \cos a = \cos\tfrac{1}{2}(n+\tfrac{1}{2})\pi\,. \tag{14}$$

Again let x be of the form $2r\pi + \frac{1}{2}\pi$; then $\cos(2x\cos^2\frac{1}{2}\varphi)$ will be replaced by $\sin(2x\cos^2\frac{1}{2}\varphi)$, and the form of $J_n(x)$ will be

$$\left[\frac{2}{\pi x}\right]^{\frac{1}{2}} \frac{1}{\Gamma(n+\frac{1}{2})}\int_0^\infty \sin t \,.\, t^{n-\frac{1}{2}} dt\,,$$

which reduces to $\left[\frac{2}{\pi x}\right]^{\frac{1}{2}} \sin\frac{1}{2}(n+\frac{1}{2})\pi$; and, at the same time, the expression for y in (12) becomes

$$y = \frac{P}{\sqrt{x}}\cos(x-a) = \frac{P}{\sqrt{x}}\sin a\,;$$

$$\therefore \qquad \sin a = \sin\tfrac{1}{2}(n+\tfrac{1}{2})\pi\,.$$

From this and (14) it follows that

$$a = \tfrac{1}{2}n\pi + \tfrac{1}{4}\pi\,;$$

whence

$$\sqrt{\frac{\pi x}{2}}\,J_n(x) = \cos(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi)\,\varphi_n(x) + \sin(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi)\,\psi_n(x)^\dagger \tag{15}$$

for all values of x and n; $\varphi_n(x)$, $\psi_n(x)$ stand for the descending series in (12).

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* Williamson's Int. Cal., Art. 124.

† While this is a known result, I have not been able to find any previous complete derivation of it from Bessel's equation, with a general determination of the constants.

An interesting application of the descending series for $J_0(x)$ is made by Sir G. G. Stokes, (Collected Papers, or Camb. Phil. Trans. Vol. IX) where he uses it to compute all the roots of the equation $J_0(x) = 0$. Lord Rayleigh applies the same method to the equation $J_1(x) = 0$ (Theory of Sound, Vol. 1, p. 273).

4. *To determine the constants P, a, in the case of $K_n(x)$.*

Use (5), substitute for $J_n(x)$ from (15), and for convenience let x be any large number; then

$$\sqrt{\frac{\pi x}{2}}\, J_n(x) = \cos\left(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right);$$

$$\therefore \quad 2\sqrt{\frac{\pi x}{2}}\, K_n(x) = \frac{\pi}{2}\left\{\sin\left(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right) + (-1)^n \sin\left(x - \tfrac{1}{4}\pi + \tfrac{1}{2}n\pi\right)\right\}.$$

Write θ for $x - \frac{1}{4}\pi - \frac{1}{2}n\pi$, then

$$\sin\left(x - \tfrac{1}{4}\pi + \tfrac{1}{2}n\pi\right) = \sin(\theta + n\pi) = (-1)^n \sin\theta,$$

since, in this case, n is integral;

$$\therefore \quad K_n(x) = \sqrt{\frac{\pi}{2x}}\sin\theta = \sqrt{\frac{\pi}{2x}}\cos\left(\theta - \tfrac{1}{2}\pi\right)$$

$$= \sqrt{\frac{\pi}{2x}}\cos\left(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi\right).$$

Comparing with (12), it follows that

$$P = \sqrt{\frac{\pi}{2}}, \qquad a = \tfrac{3}{4}\pi + \tfrac{1}{2}n\pi;$$

$$\therefore \quad \sqrt{\frac{2x}{\pi}}\, K_n(x) = \cos\left(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi\right)\varphi_n(x) + \sin\left(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi\right)\psi_n(x),^* \quad (16)$$

for any value of x and an integral value of n; $\varphi_n(x)$, $\psi_n(x)$ stand, as before, for the descending series in (12).

5. These series for $J_n(x)$, $K_n(x)$ terminate only when $2n$ is an odd integer. In other cases their terms decrease in magnitude for a certain number of such terms (depending on the values of n and x), and then they begin to increase; but the convergent part may be used for computation when x is large; "for it can be proved that the sum of any number of terms of the series differs from the true value of the function by less than the last term included."†

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[*] Equation (16) may be derived directly from (5) and (15) without previously assuming that $K_n(x)$ must be capable of taking the general form given in (12).

[†] Rayleigh, Theory of Sound, Vol. I, p. 264. Lommel, Bessel'sche Funktionen, p. 62.